Discussions: Relating to the Order of Operations in Algebra
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This might be worth while in case e is large, E_1 objectionable and E_2 and E_3 of no consequence.

As to what errors are tolerable and what objectionable, that is of course entirely a matter of circumstances.

Discussions.

I. Relating to the Order of Operations in Algebra.

By N. J. Lennes, University of Montana.

§ 1. *The Rules as Given in the Books.*

Subtraction and division are defined as the inverse operations of addition and multiplication. The commutative and associative laws of addition and multiplication are, therefore, extended in the same manner to both subtraction and division. In the case of addition alone or of multiplication alone it is agreed that, when no symbols of aggregation occur, the operations are to be performed from left to right. Thus, $a + b + c$ means $(a + b) + c$, and $a \times b \times c = (a \times b) \times c$. Without such an understanding the associative laws would have no meaning.

The Commutative and Associative Laws. In case symbols of addition and subtraction both occur (and no other symbols), it is agreed that each symbol applies only to the term immediately following it, and that the operations are to be performed from left to right.

Thus, $8 - 2 + 4 = (8 - 2) + 4 = 10$, and *not* $8 - (2 + 4) = 2$. From this usage it follows that terms connected by $+$ and $-$ signs may be *commuted*, but they may not be *associated*, except when a $+$ sign precedes the group in question. Thus, $8 + 4 - 2 = 8 + (4 - 2)$, but $8 - 4 - 2$ is not equal to $8 - (4 - 2)$.

In case the signs of multiplication and division occur with no signs of addition and subtraction intervening, and in case no symbols of aggregation are used, then it is likewise agreed (in the theoretical development in the books) that each symbol applies only to the factor (or divisor) immediately following it, and that the operations are to be performed in order from left to right.

Thus, $8 \div 2 \times 4 = (8 \div 2) \times 4 = 16$, and *not* $8 \div (2 \times 4) = 1$. As in the case of addition and subtraction, it results from this agreement that the commutative law applies to the operations of multiplication and division, while the associative law does not apply, except when the sign $\times$ precedes the group in question.

That is, $8 \div 4 \times 2 = 8 \times 2 \div 4$, but $8 \div 4 \times 2$ is not equal to $8 \div (4 \times 2)$. As remarked by Chrystal, under these conventions, the associative and commutative laws for addition and subtraction are *formally identical* with these laws for multiplication and division. (*Text Book of Algebra*, Part I, page 17.)

Following this theoretical development most of the current text-books give a rule like the following:

A series of operations involving multiplication and division alone shall be performed in the order in which they occur from left to right.

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§ 2. *Actual Usage.*

The Above Rule Contrary to Actual Usage. The rule stated above is agreed to by practically all those writers on algebra who make any mention of the matter at all. Chrystal gives a detailed development and writers on elementary algebra have in general followed him. It would, however, follow from this rule for carrying out multiplications and divisions in order from left to right, that

$$9a^2 \div 3a = (9a^2 \div 3) \times a = 3a^3.$$

But I have not been able to find a single instance where this is so interpreted. The fact is that the rule requiring the operations of multiplication and division to be carried out from left to right *in all cases,* is not followed by *anyone.* For example, in case an indicated product follows the sign $\div$ the whole product is *always* used as divisor, except in the theoretical statement of the case.

Writers meet the situation in different ways:

(*a*) Some always use the fractional form to indicate division, this being equivalent to a symbol of aggregation. Thus, $ab/cd = (ab) \div (cd)$.
(*b*) Some write out the words in full, thus: "divide this expression by that expression."
(*c*) Some use the sign $\div$ to mean that the whole product, following the sign $\div$, shall be the divisor.

The most important exception to (*c*) occurs in the development of the theory in such a text as Chrystal. In Chrystal, $\div\, u \times v$ is sometimes used to mean $(\div\, u)v$. In such cases, however, the notation $\div\, u \times v$ and not $\div\, uv$ is used.

Chrystal in one case writes $2^2 \div 3^2 \times 5^2 = (2^2 \div 3^2) \times 5^2$. (Note the sign $\times$ to indicate multiplication.) He also writes $pa/pb = pa \div (pb)$. (Note the parenthesis.) This comes nearer consistency than is usually the case. However, in no case does Chrystal write $9a^2 \div 3a$ as the equivalent of $(9a^2 \div 3) \times a$. He overcomes the difficulty by *never using the sign $\div$ with a product after it.*

The followers of Chrystal have too often blindly copied his theory, but have not taken pains, as he did, to avoid inconsistency in usage. Examples of such inconsistency in theory and usage could be multiplied *ad infinitum.* One text, which has been in very wide use, states (in developing the theory)

$$60 - 40 \div 5 \times 3 - 20 = 60 - \frac{40}{5} \times 3 - 20,$$

but on the next page we read:

$$10bc \div 12a = \frac{10bc}{12a}.$$

The Established Usage. When an indicated product follows the sign $\div$ the whole product is, by overwhelming preponderance of actual usage, to be regarded as the divisor. Hence, the true rule as to the order of operations when both

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multiplications and divisions are involved is not the one stated above, but the following:

All multiplications are to be performed first and the divisions next.

That is, $9a^2 \div 3a = 3a$ and not $3a^3$.

The multiplications may be taken in any order, but the divisions are to be taken in the order in which they occur from left to right.

That is, the associative law holds for the former but not for the latter.

Thus, $3 \times 5 \times 2 = (3 \times 5) \times 2$ or $= 3 \times (5 \times 2)$; but, $16 \div 4 \div 2 = (16 \div 4) \div 2$ and does not $= 16 \div (4 \div 2)$.

Compare the corresponding rules for addition and subtraction in § 1.

Mathematical Idioms. It might be agreed that, for the sake of simplicity and logical coherence, the past tense of the verb *to drink* should be *drinked*, but even so, English speaking people would continue to say *drank*, and not drinked. Precisely, for the same reason, all who know anything about the language of algebra regard $9a^2 \div 3a$ as equal to $3a$ and not $3a^3$, and, therefore, the rule just given is the correct one as determined by actual usage. When a mode of expression has become wide-spread, one may not change it at will. It is the business of the lexicographer and grammarian to record, not what he may think an expression should mean (no matter how far-fetched the usual or idiomatic usage may seem), but what it is *actually understood to mean by those who use it.* The language of algebra contains certain idioms and in formulating the grammar of this language we must note them. For example, that $9a^2 \div 3a$ is understood to mean $3a$ and not $3a^3$ is such an idiom. The matter is not logical but historical.

II. RELATING TO AN EXTENSION OF WILSON'S THEOREM.

BY ELIZABETH BROWN DAVIS, U. S. Naval Observatory.

From Wilson's theorem we have the congruence,

$$(p - 1)! + 1 \equiv 0, \quad (\text{mod. } p),$$

which may be written,

$$(p - 1)(p - 2)! + 1 \equiv 0, \quad (\text{mod. } p). \tag{1}$$

Subtracting (1) from $p(p - 2)! \equiv 0, \quad (\text{mod. } p)$, we have

$$(p - 2)! - 1 \equiv 0, \quad (\text{mod. } p).$$

This may be written

$$(p - 2)(p - 3)! - 1 \equiv 0, \quad (\text{mod. } p). \tag{2}$$

Subtracting (2) from $p(p - 3)! \equiv 0, \quad (\text{mod. } p)$, we have

$$2(p - 3)! + 1 \equiv 0, \quad (\text{mod. } p),$$

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